Physics 606 Exam 2 Skeleton Solution

1. (a) $$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right)u_\ell(r) = 0$$

has solutions $u_\ell = a r^{\ell+1}$ and $a r^{-\ell}$.

But $u_\ell = a r^{-\ell} \Rightarrow \int_\epsilon^a dr\, r^2 R_\ell(r)^2 = a^2 \int_\epsilon^\infty dr\, r^2 \frac{r^{-2\ell}}{r^2} = a^2 \left[\frac{r^{-2\ell+1}}{-2\ell+1}\right]$

which $\rightarrow -\infty$ as $\epsilon \rightarrow 0$, so this solution is unrenormalizable.

$\therefore$ $u_\ell \propto r^{\ell+1}$ and $R_\ell \propto r^\ell$ for $\ell \geq 1$

(b) In this case $\frac{d^2}{dr^2}(cr) = 0$ and $[V(r) - E]cr \rightarrow 0$ as $r \rightarrow 0$

so $R_\ell(r) \rightarrow$ const is a solution.

2. (a) $$\boxed{\langle\psi|H|\psi\rangle =} \sum_{\ell mn} |C_{\ell mn}|^2 \underbrace{\langle \ell m n|H|\ell m n\rangle}_{= -\frac{E_0}{n^2}}$$

$$= \frac{1}{36}\left(\frac{16}{1^2} + \frac{9}{2^2} + \frac{1}{2^2} + \frac{10}{2^2}\right)(-E_0)$$

$$= \boxed{-\frac{21}{36}E_0}$$

since $\sum_{\ell mn} |C_{\ell mn}|^2 = \frac{1}{36}(16 + 9 + 1 + 10) = 1$

so $|\psi\rangle$ is normalized

(b) Similarly $\boxed{\langle\psi|L^2|\psi\rangle = \frac{10}{9}\hbar^2}$

since $L^2|\ell m n\rangle = \ell(\ell+1)\hbar^2|\ell m n\rangle$

(c) And $L_z|\ell m n\rangle = m\hbar \Rightarrow \boxed{\langle\psi|L_z|\psi\rangle = -\frac{1}{36}\hbar}$

3. From expressions on front page for $L_\pm$ etc., after substantial algebra (in an obvious notation)

$$L_+ = \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_- = \sqrt{2}\hbar\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{\hbar}{\sqrt{2}}i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and after more substantial algebra

$$[L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y$$

or the same with L_1, L_2, L_3.

4. (a) $\langle \vec p_0 | \left[\vec r f(\vec p) - f(\vec p)\vec r \right] | \psi \rangle$

since Hermitian, $\langle \vec p_0 | f(\vec p) = \langle \vec p_0 | f(\vec p_0)$

$$= i\hbar \vec\nabla_{\vec p_0} \langle \vec p_0 | f(\vec p) | \psi \rangle - f(\vec p_0) \langle \vec p_0 | \vec r | \psi \rangle$$

$$= i\hbar \vec\nabla_{\vec p_0} \left[f(\vec p_0) \langle \vec p_0 | \psi \rangle \right] - f(\vec p_0) \left[i\hbar \vec\nabla_{\vec p_0} \langle \vec p_0 | \psi \rangle \right.$$

(b) $\vec\nabla_{\vec p_0} \left[f(\vec p_0) \langle \vec p_0 | \psi \rangle \right] = f(\vec p_0) \vec\nabla_{\vec p_0} \langle \vec p_0 | \psi \rangle + \left[\vec\nabla_{\vec p_0} f(\vec p_0) \right] \langle \vec p_0 | \psi \rangle$

so, after terms cancel,

$$\langle \vec p_0 | [\vec r, f(\vec p)] | \psi \rangle = i\hbar \left[\vec\nabla_{\vec p_0} f(\vec p_0) \right] \langle \vec p_0 | \psi \rangle$$

$$= \langle \vec p_0 | i\hbar \vec\nabla_{\vec p} f(\vec p) | \psi \rangle$$

since $\langle \vec p_0 | F(\vec p) = \langle \vec p_0 | F(\vec p_0)$, $F(\vec p) \equiv \vec\nabla_{\vec p} f(\vec p)$.

This is true for all $|\psi\rangle$, so the two operators are the same operator:

$$[\vec r, f(\vec p)] = i\hbar \vec\nabla_{\vec p} f(\vec p) .$$

(c) With $f(\vec p) = e^{-i\vec p\cdot\vec R/\hbar}$,

$$\vec r\, e^{-i\vec p\cdot\vec R/\hbar} = e^{-i\vec p\cdot\vec R/\hbar}\, \vec r + i\hbar \underbrace{\vec\nabla_{\vec p}\, e^{-i\vec p\cdot\vec R/\hbar}}_{= -\frac{i\vec R}{\hbar} e^{-i\vec p\cdot\vec R/\hbar}}$$

so $e^{i\vec p\cdot\vec R/\hbar}\, \vec r\, e^{-i\vec p\cdot\vec R/\hbar} = \vec r + \vec R$

(d) With $\vec R \to -\vec R$ above,

$$\vec r \left(e^{+i\vec p\cdot\vec R/\hbar} | \vec r_0 \rangle \right) = (\vec r_0 - \vec R) \left(e^{+i\vec p\cdot\vec R/\hbar} | \vec r_0 \rangle \right)$$

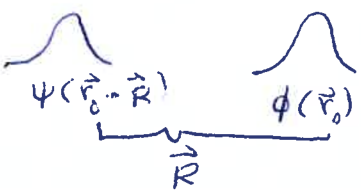
so $e^{i\vec p\cdot\vec R/\hbar} | \vec r_0 \rangle = \text{constant} \times | \vec r_0 - \vec R \rangle$.

But $\langle \vec r_0 | e^{-i\vec p\cdot\vec R/\hbar} \cdot e^{i\vec p\cdot\vec R/\hbar} | \vec r_0 \rangle = 1$

so constant = 1.

Then $\boxed{\langle \vec r_0 | e^{-i\vec p\cdot\vec R/\hbar} = \langle \vec r_0 - \vec R |}$.

(e) $\boxed{\phi(\vec r_0) =} \langle \vec r_0 | e^{-i\vec p\cdot\vec R/\hbar} | \psi \rangle = \langle \vec r_0 - \vec R | \psi \rangle = \boxed{\psi(\vec r_0 - \vec R)}$



$\to \vec r_0$ (schematically)

$\vec p$ is generator of translations